**Numerical Methods for Science and Engineering**

**Lecture Note 6**

**Interpolation & Curve Fitting**

**6.1 Introduction**

In many occasions we are given only a few discrete set of values. To study the behavior of the function through those points a technique known as **interpolation** is introduced. Polynomial is a function which is easy to handle. The method of finding a polynomial that fits a selected set of points which behaves nearly the same way as the true function will be considered.

**6.2 Polynomial Interpolation**

Given the values of a function *f*(*x*) at (*n*+1) distinct points  we can construct a **unique** polynomial of degree less than equal to *n* which satisfies the conditions



**General Form :** An *n*th degree polynomial can be taken as



To fit this polynomial to (*n*+1) set of points we have to solve (*n*+1) simultaneous equations and is very tedious.

**Newton Interpolating Polynomial :** A form which is convenient to use is suggested by Newton is



The unknown coefficients can be determined successively by substituting the set of values given. This form of representation is convenient in determining the unknown coefficients and plays an important role in the derivation of an interpolating polynomial.

**Example 6.1** : Find the polynomial of least degree which takes the values

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | −1 | 1 | 2 | 5 |
| *f*(*x*) | 9 | 3 | 6 | 39 |

**Solution:** There are four set of values given. Let the approximated polynomial be



Using the values of *x* and *f*(*x*) in turn, we get

From , we get 

From , we get  or 

From , we get  or 

From , we get or



Thus the polynomial is 





**6.3 Divided Differences**

Interpolating polynomials can be expressed in a variety of forms, and among these the Newton divided difference form is probably the convenient and efficient one.

Let the values of  corresponding to the arguments  be

.

The first divided difference for arguments  and  is defined by :



The second divided difference for arguments, and  is defined as :



Similarly higher divided differences are defined. The *n*th divided differences with (*n*+1) arguments is defined by 

**Property 1:** The divided differences are symmetric about their arguments i.e. does not depend on the order of the arguments.

**Property 2:** The *n*th divided differences of a polynomial of degree *n* is constant

**6.4 Interpolation Formula using Divided Differences**

**6.4.1 Newton Divided Difference Interpolation**

The interpolating polynomial  through the points  can be written in the Newton form as



Substituting , we have



 or 



or 

or 

Continuing the process it can be shown that 

Thus in terms of the divided differences interpolating polynomial can be written as





This is known as **Newton’s divided difference interpolation formula**.

If is a polynomial through (*n*+1) points , then the polynomial  through those points with an extra point is 

The constant *b* can be calculated by substituting 

Example 6.2

The table below gives the values of *x* and *f*(*x*):

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* : | −1 | 1 | 2 | 3 | 4 |
| *f*(*x*) : | −7 | −1 | 8 | 29 | 68 |

(i) Construct a divided-difference table for the above data.

(ii) Find the polynomial of least degree that incorporates the values in the table and find .

1. Find by linear interpolation a real root of .

(iv) Find the polynomial  that takes the values of the above table and .

**Solution:**

(i) The divided difference table for the given data is as follows:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | *f*(*x*) | f1[ ] | f2[ ] | f3[ ] | f4[ ] |
| -1 | -7 |  |  |  |  |
| 1 | -1 | 3 |  |  |  |
| 2 | 8 | 9 | 2 |  |  |
| 3 | 29 | 21 | 6 | 1 |  |
| 4 | 68 | 39 | 9 | 1 | 0 |

(ii) The needed differences are enclosed by the double lined box.

By Newton’s divided difference formula, we get

****

and ****

****

(iii) Here ****

Thus a root is in (1, 2).

From the table, we have

*x* *f*(*x*) 1DD

1 −1

1. 8 9

Thus the root is the solution of

****

or  ****

(iv) The polynomial  can be written as



where *b* is a constant.

Taking , we have

****

or 

Hence 

The required polynomial is

****

**6.4.2 Newton Backward Divided Difference Formula**

If the nodes are reordered as , the divided differences interpolating polynomial can be written as





and is called the **Newton Backward Divided Difference** formula.

**6.5 Lagrange Interpolating Polynomial**

Lagrange polynomial of degree one passing through two points  and  is written as



Lagrange polynomial of degree two passing through three points , and  is written as



Lagrange polynomial of degree three passing through four points ,,  and  is written as





In general, the Lagrange polynomial of degree *n* passing through  points ,, ⋅ ⋅ ⋅ ,  is written as





**Example 6.3**

The following table gives the values of an empirical function *f*(*x*) for certain values of *x*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | 0 | 1 | 2 | 3 |
| *f*(*x*) | −4 | −1 | 8 | 29 |

Use the Lagrange interpolation formula to estimate

1. the value of 
2. the root of the equation *f*(*x*) = 0 to 3 decimal places.

(i) Applying Lagrange’s formula, we have

****

****

and ****

****

****

****

(ii) Let . Then the root of  corresponds to . To find the root let us use the Lagrange formula in reverse order i.e. consider the polynomial in terms of *y*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *y* | −4 | −1 | 8 | 29 |
| *x* | 0 | 1 | 2 | 3 |

Then

****

When , then

****

****

****

**Exercise 6.4** The upward velocity of a rocket is given as a function of time below:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* (s) | 10 | 15 | 20 | 22.5 | 30 |
| *v*(*t*) (m/s) | 227 | 363 | 517 | 603 | 903 |

i. Construct a divided-difference table for the above data.

ii. Determine the value of the velocity at seconds using two suitable points.

iii. Determine the value of the velocity at seconds using three suitable points.

iv. Find the polynomial which passes through all the points and find .

v. Use Lagrange interpolating polynomial to estimate

a. the value of t for using two suitable points.

b. the value of *t* for using three suitable points.

vi. Write down MATLAB codes using “**polyfit(x, y, n)**” and “**polyval(p, x)**” for the following.

a. Find the polynomial of least degree that incorporates all the values in the table. and estimate the velocities corresponding to . seconds.

b. Draw the figure showing fitted polynomial and the given points.

**Solution:**

i.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* | *v*(*t*) | *v*1[ ] | *v*2[ ] | *v*3[ ] | *v*4[ ] |
| 10 | 227 |  |  |  |  |
| 15 | 363 | 27.2 |  |  |  |
| 20 | 517 | 30.8 | 0.36 |  |  |
| 22.5 | 603 | 34.4 | 0.48 | 0.0096 |  |
| 30 | 903 | 40.0 | 0.56 | 0/0043 | 0.0002 |

ii. Note that and using the relevant part of the table

|  |  |  |
| --- | --- | --- |
| *t* | *v*(*t*) |  |
| 15 | 363 |  |
| 20 | 517 | 30.8 |

we have the linear polynomial

And

iii. Note that . Thus we may choose points corresponding to . Collecting the relevant part of the table

|  |  |  |  |
| --- | --- | --- | --- |
| *t* | *v*(*t*) | *v*1[ ] | *v*2[ ] |
| 15 | 363 |  |  |
| 20 | 517 | 30.8 |  |
| 22.5 | 603 | 34.4 | 0.48 |

The polynomial with 3 points is

And

iv. Polynomial passing through all the points is

And

v. For a given *v* we need to calculate the value of *t*, so consider the Lagrange polynomial in reverse order.

a. With two points consider

and the Lagrange polynomial is

For ,

b. With three points consider

and the Lagrange polynomial is

For ,

vi. MATLAB CODES

a. >> t=[10 15 20 22.5 30];

>> v=[227 363 517 603 903];

>> pt=polyfit(t,v,4)

pt = -0.0002 0.0240 -0.4267 28.2000 -34.2000

>> t1=[17 25 30];

>> v1=polyval(pt,t1);

>> % Output value of v for t

>> t\_v =[t1',v1']

t\_v = 17.0000 421.9875

25.0000 695.8000

30.0000 903.0000

b. >> t=[10 15 20 22.5 30];

>> v=[227 363 517 603 903];

>> pt=polyfit(t,v,4);

>> t1=linspace(5,35,500); % generates 500 values

>> v1=polyval(pt,t1); % calculates values of v

>> plot(t, v,'o',t1,v1);

>> title('Graph of v against t');

>> xlabel('Time (t)');

>> ylabel('Velocity v(t)');

**Exercise 6.1**

1. The table below gives the velocity *v* at time *t*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *t(s)* | 1 | 3 | 4 | 7 |
| *v(m/s)* | 3 | 5 | 21 | 201 |

i. Construct a divided-difference table for the above data.

ii. Find the polynomial of least degree that incorporates the values in the table.

iii. Find the acceleration at time .

iv. Find the distance function when .

2. . The table below gives the values of

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | -2 | 0 | 3 | 6 | 7 |
| *f*(*x*) | 2 | -4 | -58 | 842 | 1802 |

1. Construct a divided-difference table for the above data.
2. Find the polynomial which passes through all the points of the table and find .
3. Find the polynomial  that takes the values of the above table and
4. Use Lagrange interpolating polynomial to estimate
   1. the value of using two suitable points.
   2. the value of *x* for *f* using three suitable points.
5. Write down MATLAB codes using “**polyfit(x, y, n)**” and “**polyval(p, x)**” for the following.

Find the polynomial of least degree that incorporates all the values in the table and estimate the values corresponding to .

3. The table below gives the values of

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 4 | 5 | 7 | 9 | 11 |
|  | 62 | 95 | 185 | 307 | 461 |

1. Construct a divided-difference table for the above data.
2. Find the polynomial which passes through all the points of the table and find .
3. Find the polynomial  that takes the values of the above table and
4. Use Lagrange interpolating polynomial to estimate
5. the value of using two points.
6. the value of *x* for *f* using three points.
7. Write down MATLAB codes using “**polyfit(x, y, n)**” and “**polyval(p, x)**” for the following.

Find the polynomial of least degree that incorporates all the values in the table and estimate the values corresponding to .

4. The table below gives the values of *x* and *f*(*x*):

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | -2 | -1 | 0 | 3 |
| *f*(*x*) | 12 | 14 | 10 | 22 |

1. Construct a divided-difference table for the above data.
2. Find the polynomial of least degree that incorporates the values in the table and find .
3. Given , find the polynomial  that also takes the values of the above table.
4. Use Lagrange interpolation formula to find
   1. a real root of using linear approximation.
   2. a real root of using all the points.
5. Write down MATLAB codes using “**polyfit(x, y, n)**” and “**polyval(p, x)**” to plot the figure showing fitted polynomial and the given points.

**Curve Fitting**

**6.6 Introduction**

The purpose of curve fitting is to find the parameters values of the model function that closely match the data’s. The fitted curves can be used to estimate the values of one variable corresponding to the specified values of the other variable. The method of least squares may be one of the most systematic procedure to fit a curve through the given data points. In polynomial interpolation we have considered the problem of finding polynomial of least degree which agree with the tabulated data’s. Spline interpolation is a form of polynomial interpolation where the interpolant is a piecewise polynomial called spline. This means that between two points there is a piecewise polynomial curve which joined smoothly to the neighboring curves. Cubic spline has different important applications. One of the important applications is in Computer graphics.

**6.7 Curve Fitting by Least Squares Method**

The method of least squares may be one of the most systematic procedures to fit a curve through given data points.

Consider the problem of fitting a set of *n* data points



to a curve  whose values depends on *m* parameters . The values of the function at a point depends on the values of the parameter involved. In least square method we determine a set of values of the parameter  such that the sum of the squares of the error



is minimum.

The necessary conditions for *E* to have a minimum is that



This condition gives a system of *m* equations, ***called normal equations***, in *m* unknowns.

If the parameters appear in the function in non-linear form, the normal equations become non-linear and are difficult to solve. This difficulty may be avoided if  is transformed to a form which is linear in parameters.

**Note that**. 

**Example 6.5**

Given the following set of values of *x* and *y*:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *X* | 1 | 2 | 3 | 4 | 5 | 6 |
| *Y* | 1.553 | 1.638 | 0.685 | −0.428 | −0.679 | 0.164 |

A physicist wants to approximate the data using a periodic curve .

Estimate the parameters *a* and *b* to 2 decimal places using least squares method.

**Solution**

Sum of the square deviation is



At minimum,

and 

These conditions gives





which can be rearranged as





The sum can be calculated as follows

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| ***X*** | ***y*** | **sin *x*** | **sin*2x*** | ***y* sin *x*** |
| 1 | 1.553 | 0.8415 | 0.7081 | 1.3068 |
| 2 | 1.638 | 0.9093 | 0.8268 | 1.4894 |
| 3 | 0.685 | 0.1411 | 0.0199 | 0.0967 |
| 4 | -0.428 | -0.7568 | 0.5727 | 0.3239 |
| 5 | -0.679 | -0.9589 | 0.9195 | 0.6511 |
| 6 | 0.164 | -0.2794 | 0.0781 | -0.0458 |
| Sum | 2.933 | -0.1032 | 3.1251 | 3.8221 |

The normal equations are

6 *a*− 0.1032 *b* = 2.933

−0.1032 *a* + 3.1251 *b* = 3.8221

By dividing each equation by the coefficient of *a*, we have

*a*− 0.0172 *b* = −0.4888

*a*− 30.282 *b* = −37.0359

Subtracting the equations

30.2648 *b* = 37.5247

Solving we have 



**Example 6.6**

The height of a child is measured at different ages and listed below:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* (yrs) | 3 | 6 | 9 | 12 | 15 |
| *H* (ft ) | 2.87 | 3.60 | 4.28 | 4.88 | 5.35 |

It is believed that height follows saturation growth model

1. Use a suitable substitution to reduce the above relation to a linearized form in parameters.
2. Use least square method to find the normal equation of the above data
3. Estimate, to 2 decimal places, the values of
4. Estimate the height when the child becomes 20 years old.
5. Use MATLAB function to fit the general form like

**Solution**

i. The curve is to be fitted to the given data.

The equation of the curve can be rewritten as

Taking logarithm of both sides, we get t,

which can be written in the form



where .

ii. Sum of the square deviation is



At minimum,  and 

These conditions give





which can be rearranged as





The sum can be calculated in a tabular form as shown below:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| N | T | H | X | Y | XY | X2 |
| 1 | 3 | 2.87 | 3 | 0.221 | 0.663 | 9 |
| 2 | 6 | 3.60 | 6 | -0.234 | -1.402 | 36 |
| 3 | 9 | 4.28 | 9 | -0.679 | -6.113 | 81 |
| 4 | 12 | 4.88 | 12 | -1.134 | -13.609 | 144 |
| 5 | 15 | 5.35 | 15 | -1.582 | -23.727 | 225 |
| Sum |  |  | 45 | -3.408 | -44.187 | 495 |
|  |  |  |  |  |  |  |
| Normal Equations | | |  |  |  |  |
|  | 5 | A + | 45 | B = | -3.408 |  |
|  | 45 | A + | 495 | B = | -44.187 |  |
| iii. Solutions: |  | A + | 9 | B = | -0.682 |  |
|  |  | A + | 11 | B = | -0.982 |  |
|  |  |  | -2 | B = | 0.3 |  |
|  | |  |  |  |  |  |
|  | B = | -0.150 | a3 = | 0.15 |  |  |
|  | A = | 0.668 | a2 = | 1.95 |  |  |

iv. The fitting curve is

From the equation of the curve, we get

when .

v. >> xd=[3 6 9 12 15]; % state x-values

>> yd=[2.87 3.60 4.28 4.88 5.35]; % staet y-values

Define fitting curve in terms of parameters as vector a

>> fun=@(a,xd) a(1)./(1+a(2).\*exp(-a(3).\*xd));

>> a0=[6,2,0.2]; % guess parameter values

% To fit the curve use MATLAB function **lsqcurvefit** with following syntax

>> a=lsqcurvefit(Fd,a0,xd,yd)

**Exercise 6.2**

1. Find the least square line to the following data (where are constants)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* | 0 | 2 | 4 | 5 7 | 7 |
| *v* | 6.7 | 9.2 | 11.5 | 15.6 | 19.2 |

2. Average price, *P*, of a certain type of second-hand car is believed to be related to its age,*t* years, by an equation of the form

where*a* and *b* are constants. Data from a recent newspaper give the following average price (in Taka) for used car of this type,

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *t* (yrs) | 2 | 4 | 6 | 8 |
| *P* (lac) | 20.50 | 17.25 | 14.50 | 11.75 |

(i) Estimate the values of *a* and *b* rounded to 3 significant figures.

(ii) Estimate the values of a car of this type that is 10 years old and the original new price.

3. A bowl of hot water is kept in a room of constant temperature 250C. At 5 minutes interval temperature of the water is recorded and listed as given below.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* in minute | 5 | 10 | 15 | 20 | 25 |
| *T* in 0C | 76.8 | 70.4 | 64.2 | 58.8 | 54.1 |

The law of cooling can be assumed to be of the form .

(i) Find, to 2 significant figures, the best values of *a* and *k*.

(ii) Estimate the initial temperature.

(iii) Estimate the time, to the nearest minute, when the temperature of the water in the bowl will be 500C.

4. The equation can be used for calculating the speed of a moving car, where  are constants.

The table below shows the speed of the car at various times

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *t* | 4 | 8 | 12 | 16 | 20 |
| *v* | 23.21 | 28.52 | 33.07 | 36.96 | 40.29 |

1. Estimate the values of *c* and *k* rounded to 2 significant figures.

(b) Find the time, to the nearest second, when the speed is 45 ms-1.